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Review

Complete spacelike submanifolds with parallel mean curvature vector in a semi-Riemannian space form

A. Brasil[a], Rosa M.B. Chaves[b], M. Mariano[c,*]

[a] *Departamento de Matemática, Universidade Federal do Ceará, Campus do Pici, Fortaleza - CE, CEP 60455-760, Brazil*
[b] *Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão, 1010, São Paulo - SP, CEP 05508-090, Brazil*
[c] *Departamento de Matemática, Universidade Federal do Maranhão, Campus do Bacanga, São Luís - MA, CEP 65085-580, Brazil*

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Abstract

In this work we obtain a gap theorem for spacelike submanifolds with parallel mean curvature vector in a semi-Riemannian space form.
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Contents

* Corresponding address: Universidade Federal do Maranhao, Departamento de Matematica, Avenida dos Portugueses, s/n, Cam... 68000-000 Sao Luis, Maranhao, Brazil. Tel.: +55 09821 32178211.
E-mail addresses: aldir@mat.ufc.br (A. Brasil), rosab@ime.usp.br (R.M.B. Chaves), maxwell@demat.ufma.br (M. Mariano).
URLs: http://www.mat.ufc.br (A. Brasil), http://www.ime.usp.br (R.M.B. Chaves), http://www.demat.ufma.br (M. Mariano).

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1. Introduction

Let $N_p^{n+p}(c)$ be an n-dimensional connected semi-Riemannian manifold of constant curvature c whose index is p. It is called an indefinite space form of index p. A submanifold immersed in $N_p^{n+p}(c)$ is said to be spacelike if the induced metric in M^n from that of the ambient space $N_p^{n+p}(c)$ is positive definite. Spacelike submanifolds usually appear in the study of questions related to the causality in general relativity. More precisely, level sets of a function of global time are spacelike submanifolds. Also, spacelike hypersurfaces with constant mean curvature are convenient as initial hypersurfaces for the Cauchy problem in arbitrary space time and for studying the propagation of gravitational radiation. Since Goddard's conjecture (see [11]) several papers about spacelike hypersurfaces with constant mean curvature in de Sitter space have been published (see [1,7,13,16–18,21,9]). We point out Aiyama's result [2] proving that a compact spacelike submanifold with parallel mean curvature vector and flat normal bundle in de Sitter space $S_p^{n+p}(c)$ is totally umbilical. Alias et al. in [3] obtained also some rigidity results for spacelike submanifolds with parallel mean curvature vector in pseudo-Riemannian space forms $N_p^{n+p+1}(c)$. Cheng [8], generalized the results obtained in [1] to complete spacelike submanifolds in de Sitter space $S_p^{n+p}(c)$. Li [15] extended Montiel's result in [17] for complete spacelike submanifolds with parallel mean curvature vector with two topological ends. Liu [14], characterized the complete spacelike submanifolds M^n, with parallel mean curvature vector satisfying $H^2 = 4(n-1)c/n^2$ $(c > 0)$ in de Sitter space $S_p^{n+p}(c)$. He shows that M^n is totally umbilical, or M^n is the hyperbolic cylinder in $S_p^{n+p}(c)$ or M^n has unbounded volume and positive Ricci curvature. Recently, in [6], Baek et al., obtained an optimal estimate of the squared norm of the second fundamental form for complete spacelike hypersurfaces with constant mean curvature in a locally symmetric Lorentz space satisfying some curvature conditions and characterized the totally umbilical hypersurfaces. In particular, semi-Riemannian space forms $N_p^{n+p}(c)$ are examples of locally symmetric semi-Riemannian spaces.

In this paper we extend the last result to higher codimensional spacelike submanifolds with parallel mean curvature vector in a semi-Riemannian space form $N_p^{n+p}(c)$. Moreover we extend also to spacelike submanifolds a gap theorem obtained by Brasil et al. in [7] for hypersurfaces.

In the context of submanifolds, there is a well known result of Ishihara (see [12]) that, for an n-dimensional complete maximal spacelike submanifold M^n immersed in $N_p^{n+p}(c)$, if $c \geq 0$, then M^n is totally geodesic and if $c < 0$, then $0 \leq S \leq -npc$. Thus, from now on we are going to consider only the case $H \neq 0$.

Our result can be stated as:

Theorem 1.1. *Let M^n be an n-dimensional $(n \geq 3)$ complete spacelike submanifold with parallel mean curvature vector in an $(n+p)$-dimensional indefinite space form $N_p^{n+p}(c)$. Let $Q(x) = (n-2)^2 x + 4(n-1)$ and S denote the squared norm of the second fundamental form of M.*

(1) *If $H^2 < \frac{4(n-1)}{Q(p)} c$ then $c > 0$, $S \equiv nH^2$ and M is totally umbilical.*

(2) *If $H^2 = \frac{4(n-1)}{Q(p)} c$, then $c > 0$ and either $S \equiv nH^2$ and M is totally umbilical or $\sup S = nc\frac{Q(p^2)}{Q(p)}$.*

(3) *If* $H^2 > \frac{4(n-1)}{Q(p)}c$ *and* $c < 0$*, then either* $S \equiv nH^2$ *and* M *is totally umbilical or* $nH^2 < \sup S \leq S^+$*, where*

$$S^+ = \frac{n(n-2)}{2(n-1)}\left(\frac{H^2\left[Q(p^2)+pQ(p)\right]}{2(n-2)} + |H|\sqrt{Q(p)H^2-4(n-1)c}\right) - npc.$$

(4) *If* $H^2 > \frac{4(n-1)}{Q(p)}c$ *and* $c > 0$*, then either* $S \equiv nH^2$ *and* M *is totally umbilical or* $S^- \leq \sup S \leq S^+$*, where*

$$S^- = \frac{n(n-2)}{2(n-1)}\left(\frac{H^2\left[Q(p^2)+pQ(p)\right]}{2(n-2)} - |H|\sqrt{Q(p)H^2-4(n-1)c}\right) - npc.$$

(5) *Suppose that* $\frac{4(n-1)}{Q(p)}c \leq H^2 \leq c$*, and that* M^n *is not totally umbilical. If* $\sup S = S^-$*, then the supremum is attained at a point of* M^n *if and only if the codimension* p *is equal to one and* M *is an isoparametric hypersurface with two principal curvatures.*

We observe that for the case $p = 1$ and $c = 1$, item (1) is Akutagawa's result in [1], item (2) with an additional topological hypothesis is Montiel's result [17], items (3), (4) and (5) are exactly Theorem 1.3 of [7]. For the case $p \geq 2$ and $c = 1$, as $\frac{4(n-1)}{Q(p)} < \frac{4(n-1)}{n^2}$ item (1) is contained in Cheng's result [8].

We would like to point out that the case of compact surfaces was already studied by L.J. Alias and A. Romero. They proved in [4] that the only compact spacelike surfaces in S_p^{2+p} with parallel mean curvature vector are the totally umbilical ones.

Many of the results recently obtained in the study of spacelike submanifolds with parallel mean curvature vector in semi-Riemannian manifolds have been applications of the classical Simon formula for the Laplacian of the length of the second fundamental form, suitably adapted to the semi-Riemannian context. This approach works especially well in the case where the normal bundle is Lorentzian. In particular this is our approach here. On the other hand, in [4] the authors introduced a new method to study compact spacelike submanifolds with parallel mean curvature in de Sitter spaces S_q^{n+p} with arbitrary signature $1 \leq q \leq p$ (see also [3] and [5]). This other approach is based on the use of certain integral formulas with a very clear geometric meaning, and it works well even in the case $q < p$.

2. Preliminaries

Throughout this section we will introduce some basic facts and notation that will appear in the paper. Let $N_p^{n+p}(c)$ be an indefinite space form and M^n an n-dimensional complete spacelike submanifold with parallel mean curvature vector in $N_p^{n+p}(c)$. We choose a local field of semi-Riemannian orthonormal frames $e_1, \ldots, e_{n+p}$ in $N_p^{n+p}(c)$ such that at each point of M^n, $e_1, \ldots, e_n$ span the tangent space of M^n. We use the following standard convention for indexes:

$$1 \leq A, B, C, \ldots \leq n+p, \quad 1 \leq i, j, k, \ldots \leq n, \quad n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p.$$

Let $\omega_1, \ldots, \omega_{n+p}$ be its dual frame field so that the semi-Riemannian metric of $N_p^{n+p}(c)$ is given by $ds^2 = \sum_i \omega_i^2 - \sum_\alpha \omega_\alpha^2 = \sum_A \varepsilon_A \omega_A^2$, where $\varepsilon_i = 1$ and $\varepsilon_\alpha = -1$. Then the structure equations of $N_p^{n+p}(c)$ are given by

$$d\omega_A = -\sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = -\sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{C,D} K_{ABCD}\omega_C \wedge \omega_D, \tag{2.2}$$

$$K_{ABCD} = c\varepsilon_A\varepsilon_B(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2.3}$$

Restricting those forms to M^n, we get

$$\omega_\alpha = 0, \quad n+1 \le \alpha \le n+p, \tag{2.4}$$

and from Cartan's lemma, we write

$$\omega_{\alpha i} = \sum_j h^\alpha_{ij}\omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji}. \tag{2.5}$$

Hence, we obtain the structure equations of M^n,

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.6}$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.7}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha \left(h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}\right), \tag{2.8}$$

where R_{ijkl} are the components of the curvature tensor of M^n and

$$B = \sum_\alpha h^\alpha e_\alpha = \sum_\alpha h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha, \tag{2.9}$$

is the second fundamental form of M^n.

Let S be the squared norm of the second fundamental form of M^n, h denote the mean curvature vector field of M^n and H the mean curvature of M^n, that is,

$$h = \frac{1}{n}\sum_\alpha \left(\sum_i h^\alpha_{ii}\right) e_\alpha, \quad H = |h|, \quad S = \sum_{\alpha,i,j} (h^\alpha_{ij})^2.$$

Remark 2.1. Here we are using definitions of H and S following the works of Li (see [15]), and Cheng (see [8]). But we point out that one could also use $H = \sqrt{-\langle h, h\rangle}$ and $S = -\langle B, B\rangle$, as given by Ishihara [12], Alias and Romero (see [4]) and others.

The normal curvature tensor $\{R_{\alpha\beta kl}\}$, the Ricci curvature tensor $\{R_{ik}\}$ and the scalar curvature R are expressed, respectively, as follows:

$$R_{\alpha\beta kl} = -\sum_m \left(h^\alpha_{km}h^\beta_{ml} - h^\alpha_{lm}h^\beta_{mk}\right) \tag{2.10}$$

$$R_{ik} = (n-1)c\delta_{ik} - \sum_\alpha \left(\sum_l h^\alpha_{ll}\right) h^\alpha_{ik} + \sum_{\alpha,j} h^\alpha_{ij}h^\alpha_{jk} \tag{2.11}$$

$$R = n(n-1)c + (S - n^2H^2). \tag{2.12}$$

Define the first and the second covariant derivatives of $\{h^\alpha_{ij}\}$, say $\{h^\alpha_{ijk}\}$ and $\{h^\alpha_{ijkl}\}$ by

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} - \sum_k h^\alpha_{jk}\omega_{ki} - \sum_k h^\alpha_{ik}\omega_{kj} - \sum_\beta h^\beta_{ij}\omega_{\alpha\beta}, \tag{2.13}$$

$$\sum_k h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} - \sum_m h^\alpha_{mjk}\omega_{mi} - \sum_m h^\alpha_{imk}\omega_{mj} - \sum_m h^\alpha_{ijm}\omega_{mk} - \sum_\beta h^\beta_{ijk}\omega_{\beta\alpha}. \tag{2.14}$$

Then by exterior differentiation of (2.5), we obtain the Codazzi equation

$$h^\alpha_{ijk} = h^\alpha_{ikj} = h^\alpha_{jik}, \tag{2.15}$$

and to get the following Ricci formula

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = -\sum_m h^\alpha_{im}R_{mjkl} - \sum_m h^\alpha_{jm}R_{mikl} - \sum_m h^\beta_{ij}R_{\alpha\beta kl}. \tag{2.16}$$

The Laplacian $\triangle h^\alpha_{ij}$ of h^α_{ij} is defined by $\triangle h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$. From (2.16) we have

$$\triangle h^\alpha_{ij} = \sum_k h^\alpha_{kkij} - \sum_{m,k} h^\alpha_{km}R_{mijk} - \sum_{m,k} h^\alpha_{mi}R_{mkjk} - \sum_{\beta,k} h^\beta_{ki}R_{\alpha\beta jk}. \tag{2.17}$$

Now, we assume that the mean curvature vector h of M^n is parallel, that is, $\nabla^\perp h = 0$, where $\nabla^\perp$ is the normal connection of M^n in $N^{n+p}_p(c)$, and that M^n is a complete spacelike submanifold in $N^{n+p}_p(c)$. Since we are assuming $H \neq 0$, we can choose $e_{n+1} = \frac{h}{H}$. Hence

$$\sum_k h^\alpha_{kki} = 0, \quad \omega_{\alpha,n+1} = 0, \quad H^\alpha H^{n+1} = H^{n+1}H^\alpha, \tag{2.18}$$

$$\operatorname{tr} H^{n+1} = nH, \quad \operatorname{tr} H^\alpha = 0, \quad \alpha \neq n+1 \qquad \text{and} \qquad R_{(n+1)\alpha ij} = 0 \tag{2.19}$$

where H^α denotes the matrix (h^α_{ij}). Let us define

$$\Phi^{n+1}_{ij} = h^{n+1}_{ij} - H\delta_{ij}, \quad \Phi^\alpha_{ij} = h^\alpha_{ij}, \quad \alpha \neq n+1. \tag{2.20}$$

Therefore

$$\Phi^{n+1} = H^{n+1} - HI, \quad \Phi^\alpha = H^\alpha, \quad \alpha \neq n+1, \tag{2.21}$$

where Φ^α denotes the matrix (Φ^α_{ij}). Then

$$|\Phi^{n+1}|^2 = \operatorname{tr}\left(H^{n+1}\right)^2 - nH^2, \tag{2.22}$$

$$\sum_{\alpha\neq n+1} |\Phi^\alpha|^2 = \sum_{\beta\neq n+1}\left(h^\beta_{ij}\right)^2, \tag{2.23}$$

and

$$\operatorname{tr}\left(\Phi^\alpha\right) = 0, \quad \forall\alpha. \tag{2.24}$$

Thus,

$$S = \sum_{\alpha=n+1}^{n+p} |\Phi^\alpha|^2. \tag{2.25}$$

By replacing (2.8), (2.10), (2.18) and (2.19) into (2.17) we get

$$\begin{aligned}\triangle h_{ij}^{n+1} &= nch_{ij}^{n+1} - nHc\delta_{ij} + \sum_{\beta,k,m} h_{km}^{n+1}h_{mk}^{\beta}h_{ij}^{\beta} - 2\sum_{\beta,k,m} h_{km}^{n+1}h_{mj}^{\beta}h_{ik}^{\beta} \\ &\quad + \sum_{\beta,k,m} h_{mi}^{n+1}h_{mk}^{\beta}h_{kj}^{\beta} - nH\sum_{m} h_{mi}^{n+1}h_{mj}^{n+1} + \sum_{\beta,k,m} h_{jm}^{n+1}h_{mk}^{\beta}h_{ki}^{\beta} \end{aligned} \tag{2.26}$$

and

$$\begin{aligned}\triangle h_{ij}^{\alpha} &= nch_{ij}^{n+1} + \sum_{\beta,k,m} h_{km}^{\alpha}h_{mk}^{\beta}h_{ij}^{\beta} - 2\sum_{\beta,k,m} h_{km}^{\alpha}h_{mj}^{\beta}h_{ik}^{\beta} \\ &\quad + \sum_{\beta,k,m} h_{mi}^{\alpha}h_{mk}^{\beta}h_{kj}^{\beta} - nH\sum_{m} h_{mi}^{\alpha}h_{mj}^{n+1} + \sum_{\beta,k,m} h_{jm}^{\alpha}h_{m,k}^{\beta}h_{ki}^{\beta}. \end{aligned} \tag{2.27}$$

Since

$$\frac{1}{2}\triangle S = \frac{1}{2}\sum_{\alpha,i,j}\triangle(h_{ij}^{\alpha})^2 = \sum_{\alpha,i,j} h_{ij}^{\alpha}\triangle h_{ij}^{\alpha} + \sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2,$$

from (2.26), (2.27) and (2.18) we have that

$$\begin{aligned}\frac{1}{2}\triangle S &= \sum_{\alpha,i,j,k}(h_{ijk}^{\alpha})^2 + ncS - n^2cH^2 - nH\sum_{\alpha}\operatorname{tr}(H^{n+1}(H^{\alpha})^2) \\ &\quad + \sum_{\alpha,\beta}[\operatorname{tr}(H^{\alpha}H^{\beta})]^2 + \sum_{\alpha,\beta\neq n+1} N(H^{\alpha}H^{\beta} - H^{\beta}H^{\alpha}) \end{aligned} \tag{2.28}$$

where $N(A) = \operatorname{tr}(AA^t)$, for all matrix $A = (a_{ij})$.

Now we recall a fundamental property for the generalized maximum principle due to Omori [20] and Yau [23].

Lemma 2.1. *Let M^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below on M^n. Let F be a C^2-function bounded from below on M^n. Then, for any $\varepsilon > 0$, there exists a point p in M^n such that*

$$|\nabla F(p)| < \varepsilon, \quad \Delta F(p) > -\varepsilon, \quad \inf F + \varepsilon > F(p),$$

where $|\nabla F|$ denotes the norm of the gradient of F and ΔF the Laplacian of F.

Recall also an algebraic lemma due to Okumura [19].

Lemma 2.2. *Let μ_i, $i = 1, \ldots, n$, be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, where $\beta =$ constant ≥ 0. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{2.29}$$

and the equality holds in (2.29) if and only if at least $(n-1)$ of the μ_i are equal to $\beta\sqrt{\frac{n}{n-1}}$ or $(n-1)$ of the numbers μ_i are equal to $-\beta\sqrt{\frac{n-1}{n}}$.

3. Some lemmas and the proof of Theorem 1.1

Since $H \neq 0$, from now on we are going to choose the orientation in order that $H > 0$. The following polynomial was introduced in [17] for the case $p = 1$.

$$P_{H,p}(x) = \frac{x^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}} Hx + n(c - H^2). \tag{3.1}$$

We state without proof some elementary properties of $P_{H,p}$.

Lemma 3.1. *Let $Q(x) = (n-2)^2 x + 4(n-1)$ and $P_{H,p}$ be the polynomial defined in (3.1). Then:*

(1) *If $H^2 < \frac{4(n-1)}{Q(p)} c$, then $c > 0$ and $P_{H,p}(x) > 0$ for any $x \in \mathbb{R}$.*

(2) *If $H^2 = \frac{4(n-1)}{Q(p)} c$, then $c > 0$ and the (double) root of $P_{H,p}$ is*

$$\vartheta_H^{\pm} = \frac{n(n-2)p}{\sqrt{n}} \sqrt{\frac{c}{Q(p)}}$$

so that $P_{H,p}(x) = \left(x - \frac{n(n-2)p}{\sqrt{n}} \sqrt{\frac{c}{Q(p)}}\right)^2 \geq 0$.

(3) *If $H^2 > \frac{4(n-1)}{Q(p)} c$, then $P_{H,p}$ has two real roots ϑ_H^- and ϑ_H^+ given by*

$$\vartheta_H^{\pm} = p \sqrt{\frac{n}{4(n-1)}} \left\{ (n-2)H \pm \sqrt{\frac{Q(p)H^2 - 4(n-1)c}{p}} \right\}.$$

ϑ_H^+ is always positive; on the other hand, $\vartheta_H^- < 0$ if and only if $H^2 > c$, $\vartheta_H^- = 0$, if and only if $H^2 = c$, and $\vartheta_H^- > 0$ if and only if $\frac{4(n-1)}{Q(p)} c \leq H^2 < c$.

The symmetric tensor Φ is defined by $\Phi = \sum_{i,j,\alpha} \Phi_{ij}^{\alpha} \omega_i \omega_j e_\alpha$, where Φ_{ij}^{α} was given by (2.20). Hence, from (2.25), we have that

$$S = |\Phi|^2 + nH^2. \tag{3.2}$$

We also need the following algebraic lemma, whose proof can be found in [22].

Lemma 3.2. *Let $A, B : \mathbb{R}^n \to \mathbb{R}^n$ be symmetric linear maps such that $AB - BA = 0$ and $\operatorname{tr} A = \operatorname{tr} B = 0$. Then*

$$|\operatorname{tr}(BA)^2| \leq \frac{(n-2)}{\sqrt{n(n-1)}} \operatorname{tr} A^2 \sqrt{\operatorname{tr} B^2}.$$

The equality holds if and only if $n-1$ of the eigenvalues x_i of A and the corresponding eigenvalues y_i of B satisfy

$$|x_i| = \left(\frac{\operatorname{tr} A^2}{n(n-1)}\right)^{\frac{1}{2}} \quad \text{and} \quad |y_i| = \left(\frac{\operatorname{tr} B^2}{n(n-1)}\right)^{\frac{1}{2}}, \quad x_i y_i \geq 0.$$

The following lemma was obtained by Chaves and Sousa Jr. in [10], and their result is going to appear in a forthcoming paper. The authors authorized us to present their proof here also, for completeness.

Lemma 3.3. *Let M^n be a complete spacelike submanifold with parallel mean curvature vector in an indefinite space form $N_p^{n+p}(c)$. Then the following inequality holds*

$$\frac{1}{2}\Delta|\Phi|^2 \geq |\Phi|^2\left(\frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| + n(c-H^2)\right). \tag{3.3}$$

Proof. Instead of using the second fundamental formula we will use the symmetric tensor Φ defined above, since $e_{n+1} = \frac{h}{H}$ is a parallel direction. From (2.20), we obtain also

$$\operatorname{tr}(\Phi_{n+1}^2) = \operatorname{tr}(H^{n+1})^2 - nH^2, \quad |\Phi|^2 = \sum_{\alpha,i,j}(\Phi_{ij}^\alpha)^2 = S - nH^2 \tag{3.4}$$

and

$$\operatorname{tr}(H^{n+1})^3 = \operatorname{tr}(\Phi_{n+1}^3) + 3H\operatorname{tr}(\Phi_{n+1}^2) + nH^3. \tag{3.5}$$

By replacing (2.20) and (3.5) in (2.28) we get

$$\begin{aligned}\frac{1}{2}\Delta|\Phi|^2 = &\sum_{\alpha,i,j,k}(\Phi_{ijk}^\alpha)^2 + n(c-H^2)|\Phi|^2 - nH\sum_\alpha \operatorname{tr}(\Phi_{n+1}\Phi_\alpha^2)\\ &+ \sum_{\alpha,\beta}(\operatorname{tr}\Phi_\alpha\Phi_\beta)^2 + \sum_{\alpha,\beta}N(\Phi_\alpha\Phi_\beta - \Phi_\beta\Phi_\alpha).\end{aligned} \tag{3.6}$$

By applying a Lemma 3.2 to Φ_α and Φ_{n+1} we get

$$|\operatorname{tr}(\Phi_{n+1}\Phi_\alpha^2)| \leq \frac{n-2}{\sqrt{n(n-1)}}|\Phi_{n+1}||\Phi_\alpha|^2$$

and so

$$\sum_\alpha \operatorname{tr}(\Phi_{n+1}\Phi_\alpha^2) \leq \frac{n-2}{\sqrt{n(n-1)}}|\Phi_{n+1}||\Phi|^2 \leq \frac{n-2}{\sqrt{n(n-1)}}|\Phi|^3. \tag{3.7}$$

Using the Cauchy–Schwarz inequality, it is easy to prove that

$$|\Phi|^4 \leq p\sum_\alpha(\operatorname{tr}(\Phi_\alpha^2))^2. \tag{3.8}$$

It follows from (3.6)–(3.8) that formula (3.3) holds. This completes the proof of Simon's type inequality.

Consider the positive smooth function f on M^n defined by

$$f = \frac{1}{\sqrt{1+|\Phi|^2}}.$$

We have

$$|\nabla f|^2 = -\frac{1}{4}\frac{|\Delta|\Phi|^2|^2}{(1+|\Phi|^2)^3} \tag{3.9}$$

and

$$f\Delta f = -\frac{1}{2}\frac{\Delta|\Phi|^2}{(1+|\Phi|^2)^2} + 3|\nabla f|^2. \tag{3.10}$$

From (3.3) and (3.10), we have

$$f\triangle f \le \frac{-|\Phi|^2}{(1+|\Phi|^2)^2}\left(\frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| + nc - nH^2\right) + 3|\nabla f|^2. \tag{3.11}$$

Since $\operatorname{tr} H^\alpha = 0$ and using formula (2.11),

$$\operatorname{Ric}(e_i) = (n-1)c - (\operatorname{tr} H^{n+1})h_{ik}^{n+1} + \sum_{\alpha,j}(h_{ij}^{\alpha})^2. \tag{3.12}$$

Assume that the second fundamental form of M^n with respect to e_{n+1} has been diagonalized so that the eigenvalues are λ_i^{n+1}. Then we have

$$\begin{aligned}\operatorname{Ric}(e_i) &\ge (n-1)c - nHh_{ii}^{n+1} + \sum_k \left(h_{ik}^{n+1}\right)^2\\ &= (n-1)c - nH\lambda_i^{n+1} + (\lambda_i^{n+1})^2 \ge (n-1)c - \frac{n^2H^2}{4}.\end{aligned} \tag{3.13}$$

Hence the Ricci curvature of M^n is bounded from below. Since M^n is spacelike and $f>0$, we can apply Lemma 2.1 to the function f. Hence there is a sequence of points p_k in M^n such that

$$\lim_{k\to\infty} f(p_k) = \inf f,\quad \lim_{k\to\infty}|\nabla f(p_k)| = 0,\quad \lim_{k\to\infty}\inf\triangle f(p_k) \ge 0. \tag{3.14}$$

From (3.13), we have $\inf(f)\neq 0$, so $\lim_{k\to\infty}|\Phi|^2(p_k) = \sup|\Phi|^2 < \infty$.

By replacing (3.14) into (3.11) we get

$$\frac{\sup|\Phi|^2}{(1+\sup|\Phi|^2)^2}\left(\frac{\sup|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sup|\Phi| + n(c-H^2)\right) \le 0. \tag{3.15}$$

Inequality (3.15) shows that $\sup|\Phi|^2 < \infty$.

Proof of Theorem 1.1. We first observe that $Q(p) = (n-2)^2p + 4(n-1) > 0$.
(1) If $H^2 < \frac{4(n-1)}{Q(p)}c$, then $c>0$ and

$$P_{H,p}(\sup|\Phi|) = \frac{\sup|\Phi|}{p} - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sup|\Phi| + n(c-H^2) > 0. \tag{3.16}$$

Hence, from (3.15), we have, $\sup|\Phi|^2 = 0$. Then $|\Phi|^2 = 0$ and from (3.2), $S = nH^2$, which asserts that M^n is totally umbilical.
(2) If $H^2 = \frac{4(n-1)}{Q(p)}c$, then $c>0$ and

$$P_{H,p}(\sup|\Phi|) = \left(\sup|\Phi| - \frac{n(n-2)p}{\sqrt{n}}\sqrt{\frac{c}{Q(p)}}\right)^2 \ge 0.$$

If $\left(\sup|\Phi| - \frac{n(n-2)p}{\sqrt{n}}\sqrt{\frac{c}{Q(p)}}\right)^2 > 0$, from (3.15) we have, $\sup|\Phi|^2 = 0$, that is $|\Phi|^2 = 0$ and M^n is totally umbilical. If $\sup|\Phi| = \frac{n(n-2)p}{\sqrt{n}}\sqrt{\frac{c}{Q(p)}}$, from (3.2) we have that $\sup S = nc\frac{Q(p^2)}{Q(p)}$.
(3) If $H^2 > \frac{4(n-1)}{Q(p)}c$, we have that

$$P_{H,p}(\sup|\Phi|) = (\sup|\Phi| - \vartheta_H^-)(\sup|\Phi| - \vartheta_H^+).$$

If $c < 0$ we infer that $\vartheta_H^- < 0$. Therefore, from (3.15) we have, $\sup|\Phi|^2 = 0$, in this case M^n is totally umbilical or $0 < \sup|\Phi| \le \vartheta_H^+$. Hence, from (2.25) we obtain $nH^2 < \sup S \le S^+$ where

$$S^+ = \frac{n(n-2)}{2(n-1)}\left(\frac{H^2\left[Q(p^2)+pQ(p)\right]}{2(n-2)} + H\sqrt{Q(p)H^2 - 4(n-1)c}\right) - npc.$$

(4) Suppose that $H^2 > \frac{4(n-1)}{Q(p)}c$ and $c > 0$. If $H^2 > c$, then $\vartheta_H^- < 0$ and from (3.15) we obtain $\sup|\Phi|^2 = 0$, in this case M^n is totally umbilical or $0 < \sqrt{\sup|\Phi|^2} \le \vartheta_H^+$.

When $\frac{4(n-1)}{Q(p)}c < H^2 < c$ we infer that $\vartheta_H^- > 0$. In this case, $\vartheta_H^- \le \sup|\Phi| \le \vartheta_H^+$. Therefore, from (2.25) we have that $S^- \le \sup S \le S^+$, where

$$S^- = \frac{n(n-2)}{2(n-1)}\left(\frac{H^2\left[Q(p^2)+pQ(p)\right]}{2(n-2)} - H\sqrt{Q(p)H^2 - 4(n-1)c}\right) - npc.$$

(5) If $\sup S = S^-$, then $\sup|\Phi| = \vartheta_H^-$. As $\frac{4(n-1)}{Q(p)}c \le H^2 \le c$ we infer that $\vartheta_H^- \ge 0$ and $|\Phi| \le \vartheta_H^-$. Since the graph of $P_{H,p}$ is a parabola opening upwards, we have $P_H(|\Phi|) \ge 0$ for each $|\Phi| \le \vartheta_H^-$. This fact and (3.3) imply $\Delta|\Phi|^2 \ge 0$, i.e., $|\Phi|^2$ is subharmonic function.

By inequality (3.15) and using that $\sup|\Phi| = \vartheta_H^-$, we get

$$0 \ge \sup|\Phi|^2\left(\frac{\sup|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n\,(n-1)}}H\sup|\Phi| + n(c-H^2)\right) = 0. \tag{3.17}$$

By hypothesis, $\sup|\Phi|^2$ is attained at some point of M, so we can apply the maximum principle to show that $|\Phi|^2$ is constant. Actually, $|\Phi|^2 = \sup|\Phi|^2 = \vartheta_H^-$ and Eq. (3.17) can be rewritten as

$$0 = \frac{1}{2}\Delta|\Phi|^2 \ge |\Phi|^2\left(\frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n\,(n-1)}}H|\Phi| + n(c-H^2)\right) = 0. \tag{3.18}$$

Then

$$\frac{1}{2}\Delta|\Phi|^2 = |\Phi|^2\left(\frac{|\Phi|^2}{p} - \frac{n(n-2)}{\sqrt{n\,(n-1)}}H|\Phi| + n(c-H^2)\right) = 0 \tag{3.19}$$

which means that equality holds in (3.3). As formulas (3.6)–(3.8) were crucial to proof (3.3), equality holds in formula (3.3) if and only if it holds in (3.7) and (3.8). More precisely, inequalities (3.7) and (3.8) turn into:

$$|\Phi^{n+1}||\Phi|^2 = |\Phi|^3, \tag{3.20}$$

and

$$|\Phi|^4 = p\sum_\alpha\left(\mathrm{tr}\,(\Phi^\alpha)^2\right)^2 = p\sum_\alpha\left(|\Phi^\alpha|^2\right)^2. \tag{3.21}$$

Note that $|\Phi|^2$ is a smooth function bounded from above on M^n so we can apply again Lemma 2.1 to the function $|\Phi|^2$. Therefore, taking subsequences if necessary, we get a sequence

(p_k) in M^n such that

$$\lim_{k\to\infty} |\Phi|^2(p_k) = \sup|\Phi|^2 = (\sup|\Phi|)^2;$$
$$\lim_{k\to\infty} \nabla|\Phi|^2(p_k) = 0; \tag{3.22}$$
$$\lim_{k\to\infty} \sup \Delta|\Phi|^2(p_k) \leq 0.$$

Since $|\Phi|^2 = \sum_{\alpha=n+1}^{n+p} |\Phi^\alpha|^2 < \sup|\Phi|^2 = \vartheta_H^- < \infty$, we have also that

$$\lim_{k\to\infty} |\Phi^\alpha|(p_k) = C_\alpha, \alpha \geq n+1, \tag{3.23}$$

where C_α are constants.

By (3.20) we obtain

$$\lim_{k\to\infty} \left(|\Phi^{n+1}||\Phi|^2\right)(p_k) = \lim_{k\to\infty} |\Phi|^3(p_k). \tag{3.24}$$

On one hand, by (3.21) and (3.24) we get

$$C_{n+1}\left(\sup|\Phi|\right)^2 = C_{n+1}\sup|\Phi|^2 = \sup|\Phi|^3 = \left(\sup|\Phi|\right)^3, \tag{3.25}$$

which implies that $C_{n+1} = \sup|\Phi| > 0$, since M^n is not totally umbilical.

On the other hand, using (3.22) once more, (3.21) and again (3.23), we obtain

$$\left(\sup|\Phi|^2\right)^2 = \lim_{k\to\infty} |\Phi|^4(p_k) = \lim_{k\to\infty} p\sum_\alpha \left(|\Phi^\alpha|^2\right)^2(p_k) = p\sum_\alpha C_\alpha^4. \tag{3.26}$$

In view of (3.26), we get $(p-1)C_{n+1}^4 + p\sum_{\alpha=n+2}^{n+p} C_\alpha^4 = 0$. Thus if $p \geq 2$, as all the constants C_α are non-negative, we infer that $0 = C_{n+1} = \sup|\Phi|$. This contradiction shows that $p = 1$, which finishes the proof of the Theorem. □

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